

# Free Rota-Baxter systems and a Hopf algebra structure\*

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**Abstract:** In this paper, we give a linear basis of a free Rota-Baxter system on a set by using the Gröbner-Shirshov bases method and then we obtain a Hopf algebra structure on a free Rota-Baxter system.

**Key words:** Gröbner-Shirshov basis; free Rota-Baxter system; Hopf algebra.

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## 1 Introduction

A triple  $(A; R, S)$  consisting of an associative unitary algebra  $A$  over a field  $k$  and two  $k$ -linear operators  $R, S : A \rightarrow A$  is called a Rota-Baxter system if, for any  $a, b \in A$ ,

$$R(a)R(b) = R(R(a)b + aS(b)), \quad S(a)S(b) = S(R(a)b + aS(b)).$$

Rota-Baxter system was introduced by Brzeziński in a recent paper [10], which can be viewed as an extension of the Rota-Baxter algebra of weight 0.

An associative unitary algebra  $A$  together with a  $k$ -linear operator  $P : A \rightarrow A$  is called a Rota-Baxter algebra of weight  $\lambda$ , if

$$P(x) \cdot P(y) = P(P(x) \cdot y) + P(x \cdot P(y)) + \lambda P(x \cdot y), \quad x, y \in A.$$

Here  $\lambda$  is a fixed element in the field  $k$ .

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Baxter [3] firstly studied the Rota-Baxter algebras. Some combinatoric properties of Rota-Baxter algebras were studied by Rota [24] and Cartier [13]. The constructions of free Rota-Baxter associative algebras on both commutative and noncommutative cases were given by using different methods, for example, [7, 13, 15, 16, 18–20, 24].

Gröbner bases and Gröbner-Shirshov bases were invented independently by Shirshov for ideals of free Lie algebras [25] and implicitly free associative algebras [25] (see also [4, 5]), by Hironaka [22] for ideals of the power series algebras (both formal and convergent), and by Buchberger [11] for ideals of the polynomial algebras. Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics. See, for example, the books [2, 9, 12], the papers [4, 5], and the surveys [6, 8]. In fact, Gröbner-Shirshov bases theory is a useful tool for constructing free objects for many algebra varieties. We will apply Gröbner-Shirshov bases method to construct free Rota-Baxter system generated by a set.

The Hopf algebra originated from the study of topology and has widely applications on mathematics and physics. Many classical Hopf algebras are build from free objects on various context, which include the free associative algebra and the enveloping algebra of Lie algebras. Recently, there are some Hopf algebra structures on other free objects, such as the dendriform algebras [23] and Rota-Baxter algebras [14, 17, 26]. Inspired by the ideas of the above papers, we will establish a Hopf algebra structure on the free object of Rota-Baxter system.

The paper is organized as follows. In Section 2, we give a Gröbner-Shirshov basis of a free Rota-Baxter system and then a linear basis of such algebra is obtained by Composition-Diamond lemma for associative  $\Omega$ -algebras. In Section 3, by using the construction of free Rota-Baxter system obtained in Section 2, we give a Hopf algebra structure on a free Rota-Baxter system.

## 2 Free Rota-Baxter systems

### 2.1 Gröbner-Shirshov bases for associative $\Omega$ -algebras

In this subsection, we review Gröbner-Shirshov bases theory for associative unitary  $\Omega$ -algebras. For more details, see for instance, [7, 21].

Let

$$\Omega = \bigcup_{m=1}^{\infty} \Omega_m,$$

where  $\Omega_m$  is a set of  $m$ -ary operators for any  $m \geq 1$ . For any set  $Y$ , let

$$\Omega(Y) = \bigcup_{m=1}^{\infty} \left\{ \omega^{(m)}(y_1, y_2, \dots, y_m) \mid y_i \in Y, 1 \leq i \leq m, \omega^{(m)} \in \Omega_m \right\}.$$

Let  $X$  be a set. Define  $\langle \Omega; X \rangle_0 = X^*$ , where  $X^*$  is the free monoid with the

unit 1 on the set  $X$ . Assume that we have defined  $\langle \Omega; X \rangle_{n-1}$ . Define

$$\langle \Omega; X \rangle_n = (X \cup \Omega(\langle \Omega; X \rangle_{n-1}))^*.$$

Then it is easy to see that  $\langle \Omega; X \rangle_n \subseteq \langle \Omega; X \rangle_{n+1}$  for any  $n \geq 0$ . Set

$$\langle \Omega; X \rangle = \bigcup_{n=0}^{\infty} \langle \Omega; X \rangle_n.$$

If  $u \in \langle \Omega; X \rangle$ , then  $u$  is said to be an  $\Omega$ -word on the set  $X$ . For any  $u \in X \cup \Omega(\langle \Omega; X \rangle)$ ,  $u$  is called prime. Therefore, for any  $u \in \langle \Omega; X \rangle$ ,  $u$  can be uniquely expressed in the canonical form

$$u = u_1 u_2 \cdots u_n, \quad n \geq 0,$$

where each  $u_i$  is prime. The breath of  $u$ , denoted by  $\text{bre}(u)$ , is defined to be the number  $n$ . By the depth of  $u$ , denoted by  $\text{dep}(u)$ , we mean  $\text{dep}(u) = \min\{n \mid u \in \langle \Omega; X \rangle_n\}$ .

An associative unitary  $\Omega$ -algebra over a field  $k$  is an associative unitary  $k$ -algebra  $A$  with a set of multilinear operators  $\Omega$ , where  $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$  and each  $\Omega_m$  is a set of  $m$ -ary multilinear operators on  $A$ .

Let  $k\langle \Omega; X \rangle$  be the linear space spanned by  $\langle \Omega; X \rangle$  over the field  $k$ . Then  $k\langle \Omega; X \rangle$  is a free associative unitary  $\Omega$ -algebra on  $X$ .

Let  $\star \notin X$ . By a  $\star$ - $\Omega$ -word on  $X$  we mean any expression in  $\langle \Omega; X \cup \{\star\} \rangle$  with only one occurrence of  $\star$ . Let  $\langle \Omega; X \rangle^\star$  denote the set of all  $\star$ - $\Omega$ -words on  $X$ . If  $\pi$  is a  $\star$ - $\Omega$ -word and  $s \in k\langle \Omega; X \rangle$ , then we call  $\pi|_s := \pi|_{\star \mapsto s}$  an  $s$ - $\Omega$ -word.

Now, we assume that  $\langle \Omega; X \rangle$  is equipped with a monomial order  $>$ . This means that  $>$  is a well order on  $\langle \Omega; X \rangle$  such that for any  $v, w \in \langle \Omega; X \rangle$  and  $\pi \in \langle \Omega; X \rangle^\star$ , if  $w > v$ , then  $\pi|_w > \pi|_v$ .

For any  $f \in k\langle \Omega; X \rangle$ , let  $\bar{f}$  be the leading  $\Omega$ -word of  $f$  with respect to the order  $>$ . If the coefficient of  $\bar{f}$  is 1, then we call that  $f$  is monic. We also call a set  $\mathbb{S} \subseteq k\langle \Omega; X \rangle$  monic if each  $s \in \mathbb{S}$  is monic.

Let  $f, g \in k\langle \Omega; X \rangle$  be monic. Then we define two kinds of compositions.

- (I) If  $w = \bar{f}a = b\bar{g}$  for some  $a, b \in \langle \Omega; X \rangle$  such that  $\text{bre}(w) < \text{bre}(\bar{f}) + \text{bre}(\bar{g})$ , then we call  $(f, g)_w = fa - bg$  the intersection composition of  $f$  and  $g$  with respect to the ambiguity  $w$ .
- (II) If  $w = \bar{f} = \pi|_{\bar{g}}$  for some  $\pi \in \langle \Omega; X \rangle^\star$ , then we call  $(f, g)_w = f - \pi|_{\bar{g}}$  the inclusion composition of  $f$  and  $g$  with respect to the ambiguity  $w$ .

If  $\bar{f} = \pi|_{\bar{g}}$ , then the transformation  $f \mapsto f - \alpha\pi|_{\bar{g}}$  is called the Elimination of the Leading Word (ELW) of  $f$  by  $g$ , where  $g$  is monic and  $\alpha$  is the coefficient of  $\bar{f}$ .

Let  $\mathbb{S} \subseteq k\langle \Omega; X \rangle$  be monic. The composition  $(f, g)_w$  is called trivial modulo  $(\mathbb{S}, w)$  if

$$(f, g)_w = \sum \alpha_i \pi_i|_{s_i},$$

where each  $\alpha_i \in k$ ,  $\pi_i \in \langle \Omega; X \rangle^*$ ,  $s_i \in \mathbb{S}$  and  $\pi_i |_{\overline{s_i}} < w$ . If this is the case, we write

$$(f, g)_w \equiv 0 \pmod{(\mathbb{S}, w)}.$$

In general, for any  $p$  and  $q$ ,  $p \equiv q \pmod{(\mathbb{S}, w)}$  means that  $p - q = \sum \alpha_i \pi_i |_{s_i}$ , where each  $\alpha_i \in k$ ,  $\pi_i \in \langle \Omega; X \rangle^*$ ,  $s_i \in \mathbb{S}$  and  $\pi_i |_{\overline{s_i}} < w$ .

A monic set  $\mathbb{S}$  is called a Gröbner-Shirshov basis in  $k\langle \Omega; X \rangle$  if any composition  $(f, g)_w$  of  $f, g \in \mathbb{S}$  is trivial modulo  $(\mathbb{S}, w)$ .

In fact, the proof of Composition-Diamond lemma for associative unitary  $\Omega$ -algebras is the same as the one for nonunitary  $\Omega$ -algebras in [7]. See also [21].

**Theorem 2.1** ([7], Composition-Diamond lemma for associative unitary  $\Omega$ -algebras) *Let  $\mathbb{S} \subseteq k\langle \Omega; X \rangle$  be monic,  $>$  a monomial order on  $\langle \Omega; X \rangle$  and  $Id(\mathbb{S})$  the  $\Omega$ -ideal of  $k\langle \Omega; X \rangle$  generated by  $\mathbb{S}$ . Then the following statements are equivalent:*

- (i) *The set  $\mathbb{S}$  is a Gröbner-Shirshov basis in  $k\langle \Omega; X \rangle$ .*
- (ii) *If  $f \in Id(\mathbb{S})$ , then  $\bar{f} = \pi |_{\overline{s}}$  for some  $\pi \in \langle \Omega; X \rangle^*$  and  $s \in \mathbb{S}$ .*
- (iii) *The set  $Irr(\mathbb{S}) = \{w \in \langle \Omega; X \rangle \mid w \neq \pi |_{\overline{s}}, \pi \in \langle \Omega; X \rangle^*, s \in \mathbb{S}\}$  is a  $k$ -linear basis of  $k\langle \Omega; X | \mathbb{S} \rangle := k\langle \Omega; X \rangle / Id(\mathbb{S})$ .*

## 2.2 Free Rota-Baxter systems

In this subsection, we give a Gröbner-Shirshov basis of a free Rota-Baxter system on a set and then a linear basis of such an algebra is obtained by the Composition-Diamond lemma for associative unitary  $\Omega$ -algebras.

Let  $X$  be a well-ordered set and  $\Omega = \{R, S\}$ , where both  $R$  and  $S$  are 1-array operators. Assume that  $R > S$ . If  $u \in \langle \{R, S\}, X \rangle$ , then we define  $\deg(u)$  to be the number of all occurrences of all  $x \in X$  and  $\omega \in \Omega$ . For example, if  $w = x_1 x_2 R(R(x_1)S(x_2))$ , where  $x_1, x_2 \in X$ , then  $\deg(w) = 7$ .

If  $u = u_1 u_2 \cdots u_n \in \langle \{R, S\}; X \rangle$ ,  $n \geq 1$ , where each  $u_i$  is prime, then we let

$$\text{wt}(u) = (\deg(u), \text{bre}(u), u_1, u_2, \dots, u_n).$$

Define the Deg-lex order  $>_{\text{Dl}}$  on  $\langle \{R, S\}; X \rangle$  as follows. For any  $u = u_1 u_2 \cdots u_n$ ,  $v = v_1 v_2 \cdots v_m \in \langle \{R, S\}; X \rangle$ , where  $u_i, v_j$  are prime, define

$$u >_{\text{Dl}} v \text{ if } \text{wt}(u) > \text{wt}(v) \text{ lexicographically,}$$

where  $u_i > v_i$  if  $\deg(u_i) > \deg(v_i)$  or  $\deg(u_i) = \deg(v_i)$  and one of the following conditions holds:

- (a)  $u_i, v_i \in X$  and  $u_i > v_i$ ;
- (b)  $u_i \in \bigcup_{Q \in \{R, S\}} Q(\Phi_\infty(X))$  and  $v_i \in X$ ;

- (c)  $u_i = \omega(u'_i), v_i = \theta(v'_i), \omega, \theta \in \{R, S\}$  and  
 $(\omega, u'_i) > (\theta, v'_i)$  lexicographically.

It is easy to see that  $>_{\text{D1}}$  is a monomial order on  $\langle \{R, S\}; X \rangle$ .

Let  $k\langle \{R, S\}, X \rangle$  be the free associative unitary  $\{R, S\}$ -algebra generated by the set  $X$ .

**Theorem 2.2** *With the order  $>_{\text{D1}}$  on  $\langle \{R, S\}; X \rangle$ , the set*

$$\mathbb{S} = \left\{ \begin{array}{l} R(u)R(v) - R(R(u)v) - R(uS(v)) \\ S(u)S(v) - S(R(u)v) - S(uS(v)) \end{array} \middle| u, v \in \langle \{R, S\}, X \rangle \right\}$$

*is a Gröbner-Shirshov basis in  $k\langle \{R, S\}, X \rangle$ .*

**Proof.** Let

$$\begin{aligned} f(u, v) &= R(u)R(v) - R(R(u)v) - R(uS(v)), \\ h(u, v) &= S(u)S(v) - S(R(u)v) - S(uS(v)), \end{aligned}$$

where  $u, v \in \langle \{R, S\}, X \rangle$ . All the possible compositions of  $\{R, S\}$ -polynomials in  $\mathbb{S}$  are listed as below:

$$\begin{aligned} &(f(u, v), f(v, w))_{w_1}, \quad w_1 = R(u)R(v)R(w), \\ &(f(\pi|_{R(u)R(v)}, w), f(u, v))_{w_2}, \quad w_2 = R(\pi|_{R(u)R(v)})R(w), \\ &(f(u, \pi|_{R(v)R(w)}), f(v, w))_{w_3}, \quad w_3 = R(u)R(\pi|_{R(v)R(w)}), \\ &(f(\pi|_{S(u)S(v)}, w), h(u, v))_{w_4}, \quad w_4 = R(\pi|_{S(u)S(v)})R(w), \\ &(f(u, \pi|_{S(v)S(w)}), h(v, w))_{w_5}, \quad w_5 = R(u)R(\pi|_{S(v)S(w)}), \\ &(h(u, v), h(v, w))_{w_6}, \quad w_6 = S(u)S(v)S(w), \\ &(h(\pi|_{S(u)S(v)}, w), h(u, v))_{w_7}, \quad w_7 = S(\pi|_{S(u)S(v)})S(w), \\ &(h(u, \pi|_{S(v)S(w)}), h(v, w))_{w_8}, \quad w_8 = S(u)S(\pi|_{S(v)S(w)}), \\ &(h(\pi|_{R(u)R(v)}, w), f(u, v))_{w_9}, \quad w_9 = S(\pi|_{R(u)R(v)})S(w), \\ &(h(u, \pi|_{R(v)R(w)}), f(v, w))_{w_{10}}, \quad w_{10} = S(u)S(\pi|_{R(v)R(w)}). \end{aligned}$$

We check that all the compositions are trivial. Here, we just check one as an example.

$$\begin{aligned} &(f(u, v), f(v, w))_{w_1} \\ &= f(u, v)R(w) - R(u)f(v, w) \\ &= -(R(R(u)v) + R(uS(v)))R(w) + R(u)(R(R(v)w) + R(vS(w))) \\ &\equiv -R(R(R(u)v)w) - R(R(u)vS(w)) - R(R(uS(v))w) - R(uS(v)S(w)) \\ &\quad + R(R(u)R(v)w) + R(uS(R(v)w)) + R(R(u)vS(w)) + R(uS(vS(w))) \\ &\equiv -R(R(R(u)v)w) - R(R(uS(v))w) - R(uS(R(v)w)) - R(uS(vS(w))) \\ &\quad + R(uS(R(v)w)) + R(uS(vS(w))) + R(R(R(u)v)w) + R(R(uS(v))w) \\ &\equiv 0 \pmod{\mathbb{S}, w_1}. \end{aligned}$$

□

**Definition 2.3** ([15]) Let  $Y, Z \subseteq \langle \{R, S\}, X \rangle$ . Define the alternating product of  $Y$  and  $Z$  with respect to  $Q \in \{R, S\}$  by

$$\begin{aligned} \Lambda_X^Q(Y, Z) = & \left( \bigcup_{n \geq 1} (YQ(Z))^n \right) \bigcup \left( \bigcup_{n \geq 0} (YQ(Z))^n Y \right) \\ & \bigcup \left( \bigcup_{n \geq 1} (Q(Z)Y)^n \right) \bigcup \left( \bigcup_{n \geq 0} (Q(Z)Y)^n Q(Z) \right). \end{aligned}$$

We define  $\Phi_n$  and  $\Gamma_n$  recursively as follows. For  $n = 0$ , define

$$\Phi_0 = X^* = \Gamma_0.$$

Assume that we have defined  $\Phi_{n-1}$  and  $\Gamma_{n-1}$ . Define

$$\Phi_n = \Lambda_X^R(\Gamma_{n-1} \setminus \{1\}, \Gamma_{n-1}) \cup \{1\}, \quad \Gamma_n = \Lambda_X^S(\Phi_n \setminus \{1\}, \Phi_n) \cup \{1\}.$$

Let

$$\Phi_\infty(X) = \bigcup_{n \geq 0} \Phi_n, \quad \Gamma_\infty(X) = \bigcup_{n \geq 0} \Gamma_n.$$

It is easy to see that  $\Phi_\infty(X) = \Gamma_\infty(X)$ . The elements of  $\Phi_\infty(X)$  are called Rota-Baxter system words.

Note that  $RS(X) = k\langle \{R, S\}, X | \mathbb{S} \rangle$  is a free Rota-Baxter system on the set  $X$ , where

$$\mathbb{S} = \left\{ \begin{array}{l} R(u)R(v) - R(R(u)v) - R(uS(v)) \\ S(u)S(v) - S(R(u)v) - S(uS(v)) \end{array} \mid u, v \in \langle \{R, S\}, X \rangle \right\}.$$

**Theorem 2.4** The set  $\text{Irr}(\mathbb{S}) = \Phi_\infty(X)$  is a linear basis of the free Rota-Baxter system  $RS(X)$ .

**Proof.** By Theorems 2.1 and 2.2, we can obtain the result.  $\square$

By using ELWs, we have the following algorithm, which is an algorithm to compute the product of two Rota-Baxter system words in the free Rota-Baxter system  $RS(X)$ .

**Algorithm 2.5** Let  $w, v \in \Phi_\infty(X)$ . We define  $w \diamond v$  by induction on  $n = \text{dep}(w) + \text{dep}(v)$ .

(a) If  $n = 0$ , then  $u, v \in X^*$  and  $w \diamond v = wv$ ;

(b) If  $n \geq 1$ , there are two cases to consider:

(i) If  $\text{bre}(w) = \text{bre}(v) = 1$ , then

$$w \diamond v = \begin{cases} Q(R(\tilde{w}) \diamond \tilde{v} + \tilde{w} \diamond S(\tilde{v})), & \text{if } w = Q(\tilde{w}), v = Q(\tilde{v}), Q \in \{R, S\}, \\ wv, & \text{otherwise.} \end{cases}$$

(ii) If  $\text{bre}(w) > 1$  or  $\text{bre}(v) > 1$  and assume that  $w = w_1 w_2 \cdots w_t$  and  $v = v_1 v_2 \cdots v_l$ , where  $u_i$  and  $v_j$  are prime, then

$$w \diamond v = w_1 w_2 \cdots w_{t-1} (w_t \diamond v_1) v_2 \cdots v_l.$$

### 3 A Hopf algebra structure on free Rota-Baxter system

In this section, similar to the Hopf algebra structure on free Rota-Baxter algebra given by Gao, Guo and Zhang [17], we establish a Hopf algebra structure on the free Rota-Baxter system  $RS(X)$ .

#### 3.1 A bialgebra structure

In this subsection, we give a bialgebra structure on the free Rota-Baxter system  $RS(X)$ .

A  $k$ -bialgebra is a quintuple  $(H, \mu, u, \Delta, \varepsilon)$ , where  $(H, \mu, u)$  is a  $k$ -algebra and  $(H, \Delta, \varepsilon)$  is a  $k$ -coalgebra such that  $\Delta : H \rightarrow H \otimes H$  and  $\varepsilon : H \rightarrow k$  are algebra homomorphisms.

Define  $\Delta : RS(X) \rightarrow RS(X) \otimes RS(X)$ , where for any  $w \in \Phi_\infty(X)$ ,  $\Delta(w)$  is defined by induction on  $\text{dep}(w)$ :

If  $\text{dep}(w) = 0$ , then we define

$$\Delta(w) = \begin{cases} 1 \otimes 1, & \text{if } w = 1, \\ 1 \otimes x + x \otimes 1, & \text{if } w = x \in X, \end{cases}$$

and

$$\Delta(w) = \Delta(x_1) \diamond \Delta(x_2) \diamond \cdots \diamond \Delta(x_m),$$

where  $w = x_1 x_2 \cdots x_m$ ,  $m \geq 2$  with each  $x_i \in X$ .

Assume that  $\Delta(w)$  has been defined for any  $w \in \Phi_\infty(X)$  with  $\text{dep}(w) \leq n$ . Let  $w \in \Phi_\infty(X)$  with  $\text{dep}(w) = n + 1$ . If  $\text{bre}(w) = 1$ , define

$$\Delta(w) = \begin{cases} R(\tilde{w}) \otimes 1 + (\text{id} \otimes R)\Delta(\tilde{w}), & \text{if } w = R(\tilde{w}), \\ R(\tilde{w}) \otimes 1 + (\text{id} \otimes S)\Delta(\tilde{w}), & \text{if } w = S(\tilde{w}), \end{cases}$$

where  $\Delta(\tilde{w})$  is defined by induction hypothesis. If  $\text{bre}(w) > 1$ , say  $w = w_1 w_2 \cdots w_m$ , where each  $w_i$  is prime, define

$$\Delta(w) = \Delta(w_1) \diamond \Delta(w_2) \diamond \cdots \diamond \Delta(w_m).$$

Define  $u : k \rightarrow RS(X)$ ,  $1_k \mapsto 1$  and  $\varepsilon : RS(X) \rightarrow k$  by

$$\varepsilon(w) = \begin{cases} 1_k, & \text{if } w = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $1_k$  is the unit of the field  $k$ .

It is easy to see that  $(RS(X), \diamond, u)$  is a unitary  $k$ -algebra.

**Lemma 3.1** *For any  $w, v \in RS(X)$ , we have*

$$\Delta(w \diamond v) = \Delta(w) \diamond \Delta(v).$$

**Proof.** It is sufficient to prove that  $\Delta(w \diamond v) = \Delta(w) \diamond \Delta(v)$  for any  $w, v \in \Phi_\infty(X)$ . Induction on  $(m, n)$ , where  $m = \text{dep}(w) + \text{dep}(v)$  and  $n = \text{bre}(w) + \text{bre}(v)$ .

If  $(m, n) = (0, 1)$ , then  $w \in X$ ,  $v = 1$  or  $w = 1$ ,  $v \in X$ . Thus  $\Delta(w \diamond v) = \Delta(w) \diamond \Delta(v)$ .

Assume that we have proved  $\Delta(w \diamond v) = \Delta(w) \diamond \Delta(v)$  for all  $u, v \in \Phi_\infty(X)$  with  $(m, n) < (p, q)$ .

Let  $u, v \in \Phi_\infty(X)$  with  $(\text{dep}(w) + \text{dep}(v), \text{bre}(w) + \text{bre}(v)) = (p, q)$ . There are two cases to consider.

Case 1. If  $q = 1$ , i.e.  $w = 1$  or  $v = 1$ , then by definition, the result is true.

Case 2. If  $q = 2$ , then we have two cases.

(i) If  $(w, v) \notin \bigcup_{Q \in \{R, S\}} Q(\Phi_\infty(X)) \times Q(\Phi_\infty(X))$ , then  $w \diamond v = wv \in \Phi_\infty(X)$  and the result is true by the definition of  $\Delta$ .

(ii) If  $(w, v) \in \bigcup_{Q \in \{R, S\}} Q(\Phi_\infty(X)) \times Q(\Phi_\infty(X))$ , then  $w = R(w')$  and  $v = R(v')$  or  $w = S(w')$  and  $v = S(v')$ . Using the Sweedler notation, we can write

$$\Delta(w') = \sum_{(w')} w'_{(1)} \otimes w'_{(2)}, \quad \Delta(v') = \sum_{(v')} v'_{(1)} \otimes v'_{(2)}.$$



(a) If  $w = R(w')$  and  $v = R(v')$ , then

$$\begin{aligned}
& \Delta(R(w') \diamond R(v')) \\
= & \Delta(R(R(w') \diamond v' + w' \diamond S(v'))) \\
= & R(R(w') \diamond v' + w' \diamond S(v')) \otimes 1 + (\text{id} \otimes R)\Delta(R(w') \diamond v' + w' \diamond S(v')) \\
= & (w \diamond v) \otimes 1 + (\text{id} \otimes R)\Delta(R(w') \diamond v' + w' \diamond S(v')) \\
= & (w \diamond v) \otimes 1 + (\text{id} \otimes R)(\Delta(R(w')) \diamond \Delta(v') + \Delta(w') \diamond \Delta(S(v'))) \\
= & (w \diamond v) \otimes 1 + (\text{id} \otimes R)((R(w') \otimes 1 + (\text{id} \otimes R)\Delta(w')) \diamond \Delta(v') \\
& + \Delta(w') \diamond (R(v') \otimes 1 + (\text{id} \otimes S)\Delta(w'))) \\
= & (w \diamond v) \otimes 1 + (\text{id} \otimes R)\left(\sum_{(v')} (R(w') \diamond v'_{(1)}) \otimes v'_{(2)} + \sum_{(w')} (w'_{(1)} \diamond R(v')) \otimes w'_{(2)} \right. \\
& \left. + \sum_{(w'), (v')} (w'_{(1)} \diamond v'_{(1)}) \otimes (R(w'_{(2)}) \diamond v'_{(2)} + w'_{(2)} \diamond S(v'_{(2)}))\right) \\
= & (w \diamond v) \otimes 1 + \sum_{(v')} (R(w') \diamond v'_{(1)}) \otimes R(v'_{(2)}) + \sum_{(w')} (w'_{(1)} \diamond R(v')) \otimes R(w'_{(2)}) \\
& + \sum_{(w'), (v')} (w'_{(1)} \diamond v'_{(1)}) \otimes (R(w'_{(2)}) \diamond R(v'_{(2)})) \\
= & (R(w') \otimes 1 + \sum_{(w')} w'_{(1)} \otimes R(w'_{(2)})) \diamond (R(v') \otimes 1 + \sum_{(v')} v'_{(1)} \otimes R(v'_{(2)})) \\
= & (R(w') \otimes 1 + (\text{id} \otimes R)(\sum_{(w')} w'_{(1)} \otimes w'_{(2)})) \diamond (R(v') \otimes 1 + (\text{id} \otimes R)(\sum_{(v')} v'_{(1)} \otimes v'_{(2)})) \\
= & (R(w') \otimes 1 + (\text{id} \otimes R)\Delta(w')) \diamond (R(v') \otimes 1 + (\text{id} \otimes R)\Delta(v')) \\
= & \Delta(w) \diamond \Delta(v).
\end{aligned}$$

(b) If  $w = S(w')$  and  $v = S(v')$ , then

$$\begin{aligned}
& \Delta(S(w') \diamond S(v')) \\
&= \Delta(S(R(w') \diamond v' + w' \diamond S(v'))) \\
&= R(R(w') \diamond v' + w' \diamond S(v')) \otimes 1 + (\text{id} \otimes S)\Delta(R(w') \diamond v' + w' \diamond S(v')) \\
&= (R(w') \diamond R(v')) \otimes 1 + (\text{id} \otimes S)(\Delta(R(w')) \diamond \Delta(v') + \Delta(w') \diamond \Delta(S(v'))) \\
&= (R(w') \diamond R(v')) \otimes 1 + (\text{id} \otimes S)((R(w') \otimes 1 + (\text{id} \otimes R)\Delta(w')) \diamond \Delta(v') \\
&\quad + \Delta(w') \diamond (R(v') \otimes 1 + (\text{id} \otimes S)\Delta(w'))) \\
&= (R(w') \diamond R(v')) \otimes 1 + (\text{id} \otimes S)\left(\sum_{(v')} (R(w') \diamond v'_{(1)}) \otimes v'_{(2)} + \sum_{(w')} (w'_{(1)} \diamond R(v')) \otimes w'_{(2)}\right. \\
&\quad \left.+ \sum_{(w'), (v')} (w'_{(1)} \diamond v'_{(1)}) \otimes (R(w'_{(2)}) \diamond v'_{(2)} + w'_{(2)} \diamond S(v'_{(2)}))\right) \\
&= (R(w') \diamond R(v')) \otimes 1 + \sum_{(v')} (R(w') \diamond v'_{(1)}) \otimes S(v'_{(2)}) + \sum_{(w')} (w' \diamond R(v')) \otimes S(w'_{(2)}) \\
&\quad + \sum_{(w'), (v')} (w'_{(1)} \diamond v'_{(1)}) \otimes (S(w'_{(2)}) \diamond S(v'_{(2)})) \\
&= (R(w') \otimes 1 + (\text{id} \otimes S)\Delta(w')) \diamond (R(v') \otimes 1 + (\text{id} \otimes S)\Delta(v')) \\
&= \Delta(w) \diamond \Delta(v).
\end{aligned}$$

Case 3. If  $q > 2$ , say  $w = w_1 w_2 \cdots w_t$  and  $v = v_1 v_2 \cdots v_l$ , then we have two cases to consider.

(i) If  $(w_t, v_1) \notin \bigcup_{Q \in \{R, S\}} Q(\Phi_\infty(X)) \times Q(\Phi_\infty(X))$ , then  $w \diamond v = wv \in \Phi_\infty(X)$  and the result is true by the definition of  $\Delta$ .

(ii) If  $(w_t, v_1) \in \bigcup_{Q \in \{R, S\}} Q(\Phi_\infty(X)) \times Q(\Phi_\infty(X))$ , then by Case 2 or by induction, we have  $\Delta(w_t \diamond v_1) = \Delta(w_t) \diamond \Delta(v_1)$ . By the definition of  $\diamond$ , we have  $w_t \diamond v_1 = \sum u_i$ , where  $\text{bre}(u_i) = 1$ . Therefore

$$\begin{aligned}
\Delta(w \diamond v) &= \Delta(w_1 \diamond w_2 \diamond \cdots \diamond (w_t \diamond v_1) \diamond v_2 \diamond \cdots \diamond v_l) \\
&= \sum \Delta(w_1 \diamond w_2 \diamond \cdots \diamond u_i \diamond v_2 \diamond \cdots \diamond v_l) \\
&= \sum \Delta(w_1) \diamond \Delta(w_2) \diamond \cdots \diamond \Delta(u_i) \diamond \Delta(v_2) \diamond \cdots \diamond \Delta(v_l) \\
&= \Delta(w_1) \diamond \Delta(w_2) \diamond \cdots \diamond \left(\sum \Delta(u_i)\right) \diamond \Delta(v_2) \diamond \cdots \diamond \Delta(v_l) \\
&= \Delta(w_1) \diamond \Delta(w_2) \diamond \cdots \diamond \Delta(w_t) \diamond \Delta(v_1) \diamond \Delta(v_2) \diamond \cdots \diamond \Delta(v_l) \\
&= \Delta(w) \diamond \Delta(v).
\end{aligned}$$

This completes the proof by induction.  $\square$

**Lemma 3.2** For any  $w, v \in RS(X)$ , we have  $\varepsilon(w \diamond v) = \varepsilon(w) \diamond \varepsilon(v)$ .

**Proof.** It is easy to prove the result.  $\square$

**Lemma 3.3** The triple  $(RS(X), \Delta, \varepsilon)$  is a coalgebra.

**Proof.** It is sufficient to prove the coassociativity and the counicity for  $w \in \Phi_\infty(X)$ .

We check the coassociativity by induction on  $(m, n)$  for  $w \in \Phi_\infty(X)$ , where  $m = \text{dep}(w)$  and  $n = \text{bre}(w)$ .

If  $(m, n) = (0, 0)$ , then  $w = 1$  and  $(\Delta \otimes \text{id})\Delta(w) = 1 \otimes 1 \otimes 1 = (\text{id} \otimes \Delta)\Delta(w)$ .

Assume that the result is true for any  $w$  with  $(\text{dep}(w), \text{bre}(w)) < (p, q)$ .

Let  $w \in \Phi_\infty(X)$  with  $(\text{dep}(w), \text{bre}(w)) = (p, q) > (0, 0)$ . Then we have two cases to consider.

Case 1. If  $q = 1$ , then  $w = x \in X$  or  $w = R(w')$  or  $w = S(w')$ .

Subcase 1. If  $w = x \in X$ , then

$$(\Delta \otimes \text{id})\Delta(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x = (\text{id} \otimes \Delta)\Delta(x).$$

Subcase 2. If  $w = S(w')$ , then

$$\begin{aligned} & (\text{id} \otimes \Delta)\Delta(S(w')) \\ &= (\text{id} \otimes \Delta)(R(w') \otimes 1 + (\text{id} \otimes S)\Delta(w')) \\ &= R(w') \otimes 1 \otimes 1 + (\text{id} \otimes (\Delta S))\Delta(w') \\ &= R(w') \otimes 1 \otimes 1 + (\text{id} \otimes R)\Delta(w') \otimes 1 + (\text{id} \otimes \text{id} \otimes S)(\text{id} \otimes \Delta)\Delta(w') \\ &= R(w') \otimes 1 \otimes 1 + (\text{id} \otimes R)\Delta(w') \otimes 1 + (\text{id} \otimes \text{id} \otimes S)(\Delta \otimes \text{id})\Delta(w') \\ &= R(w') \otimes 1 \otimes 1 + (\text{id} \otimes R)\Delta(w') \otimes 1 + (\Delta \otimes S)\Delta(w') \\ &= (R(w') \otimes 1 + (\text{id} \otimes R)\Delta(w')) \otimes 1 + (\Delta \otimes S)\Delta(w') \\ &= \Delta(R(w')) \otimes 1 + (\Delta \otimes S)\Delta(w') \\ &= (\Delta \otimes \text{id})(R(w') \otimes 1 + (\text{id} \otimes S)\Delta(w')) \\ &= (\Delta \otimes \text{id})\Delta(S(w')). \end{aligned}$$

Subcase 3. If  $w = R(w')$ , then similar to the Subcase 2, we have

$$(\Delta \otimes \text{id})\Delta(w) = (\text{id} \otimes \Delta)\Delta(w).$$

Case 2. If  $q > 1$ , then we can let  $w = w_1 w_2$  with  $w_1, w_2 \neq 1$ . Let us denote

$$\Delta(w_1) = \sum_{(w_1)} w_{1(1)} \otimes w_{1(2)}, \quad \Delta(w_2) = \sum_{(w_2)} w_{2(1)} \otimes w_{2(2)}.$$

Then

$$\begin{aligned} (\text{id} \otimes \Delta)\Delta(w) &= (\text{id} \otimes \Delta)\Delta(w_1 \diamond w_2) \\ &= (\text{id} \otimes \Delta)(\Delta(w_1) \diamond \Delta(w_2)) \\ &= \sum_{(w_1)} \sum_{(w_2)} (w_{1(1)} \diamond w_{2(1)}) \otimes \Delta(w_{1(2)} \diamond w_{2(2)}) \\ &= \sum_{(w_1)} \sum_{(w_2)} (w_{1(1)} \diamond w_{2(1)}) \otimes (\Delta(w_{1(2)}) \diamond \Delta(w_{2(2)})) \\ &= \sum_{(w_1)} \sum_{(w_2)} (w_{1(1)} \otimes \Delta(w_{1(2)}) \diamond (w_{2(1)} \otimes \Delta(w_{2(2)})). \end{aligned}$$

Similarly, we have

$$(\Delta \otimes \text{id}) \triangle (w) = \sum_{(w_1)} \sum_{(w_2)} (\Delta(w_{1(1)}) \otimes w_{1(2)}) \diamond (\Delta(w_{2(1)}) \otimes w_{2(2)}).$$

By induction hypotheses, we have

$$(\Delta \otimes \text{id})\Delta(w_1) = (\text{id} \otimes \Delta)\Delta(w_1), \quad (\Delta \otimes \text{id})\Delta(w_2) = (\text{id} \otimes \Delta)\Delta(w_2).$$

Therefore,

$$(\Delta \otimes \text{id})\Delta(w) = (\text{id} \otimes \Delta)\Delta(w).$$

This completes the proof of coassociativity by induction.

We also use induction on  $(m, n)$  for  $w \in \Phi_\infty(X)$ , where  $m = \text{dep}(w)$  and  $n = \text{bre}(w)$  to check the counicity conditions:

$$(\varepsilon \otimes \text{id})\Delta(w) = \beta_l(w), \quad (\text{id} \otimes \varepsilon)\Delta(w) = \beta_r(w),$$

where

$$\beta_l(w) : RS(X) \rightarrow k \otimes RS(X), w \mapsto 1_k \otimes w,$$

$$\beta_r(w) : RS(X) \rightarrow RS(X) \otimes k, w \mapsto w \otimes 1_k.$$

If  $(m, n) = (0, 0)$ , then it is easy to see that the result is true.

Assume that the result is true for any  $w$  with  $(m, n) < (p, q)$ .

Let  $w \in \Phi_\infty(X)$  with  $(\text{dep}(w), \text{bre}(w)) = (p, q) > (0, 0)$ .

Case I. If  $q = \text{bre}(w) = 1$ , then  $w = x \in X$  or  $w = R(w')$  or  $w = S(w')$ .

Subcase I-1. If  $w = x \in X$ , then

$$(\varepsilon \otimes \text{id})\Delta(x) = 1_k \otimes x = \beta_l(x).$$

Subcase I-2. If  $w = S(w')$ , then

$$\begin{aligned} (\varepsilon \otimes \text{id})\Delta(w) &= (\varepsilon \otimes \text{id})\Delta(S(w')) \\ &= (\varepsilon \otimes \text{id})(R(w') \otimes 1 + (\text{id} \otimes S)\Delta(w')) \\ &= (\varepsilon \otimes \text{id})(\text{id} \otimes S)\Delta(w') \\ &= (\text{id} \otimes S)(\varepsilon \otimes \text{id})\Delta(w') \\ &= (\text{id} \otimes S)(1_k \otimes w') = 1_k \otimes S(w') = \beta_l(w). \end{aligned}$$

Subcase I-3. If  $w = R(w')$ , then similar to the Subcase I-2,  $(\varepsilon \otimes \text{id})\Delta(R(w')) = \beta_l(R(w'))$ .

Case II. If  $q = \text{bre}(w) > 1$ , then we can let  $w = w_1 w_2$  with  $w_1, w_2 \neq 1$ . Let

$$\Delta(w_1) = \sum_{(w_1)} w_{1(1)} \otimes w_{1(2)}, \quad \Delta(w_2) = \sum_{(w_2)} w_{2(1)} \otimes w_{2(2)}.$$

By induction hypotheses, we have

$$(\varepsilon \otimes \text{id})\Delta(w_1) = \beta_l(w_1), \quad (\varepsilon \otimes \text{id})\Delta(w_2) = \beta_l(w_2).$$

That is

$$\sum_{(w_1)} \varepsilon(w_{1(1)}) \otimes w_{1(2)} = 1_k \otimes w_1, \quad \sum_{(w_2)} \varepsilon(w_{2(1)}) \otimes w_{2(2)} = 1_k \otimes w_2.$$

Thus

$$\begin{aligned} (\varepsilon \otimes \text{id})\Delta(w) &= (\varepsilon \otimes \text{id})\Delta(w_1 \diamond w_2) \\ &= (\varepsilon \otimes \text{id})(\Delta(w_1) \diamond \Delta(w_2)) \\ &= \sum_{(w_1), (w_2)} \varepsilon(w_{1(1)} \diamond w_{2(1)}) \otimes (w_{1(2)} \diamond w_{2(2)}) \\ &= \sum_{(w_1), (w_2)} (\varepsilon(w_{1(1)})\varepsilon(w_{2(1)})) \otimes (w_{1(2)} \diamond w_{2(2)}) \\ &= \sum_{(w_1), (w_2)} (\varepsilon(w_{1(1)}) \otimes w_{1(2)})(\varepsilon(w_{2(1)}) \diamond w_{2(2)}) \\ &= (1_k \otimes w_1)(1_k \otimes w_2) = 1_k \otimes (w_1 \diamond w_2) = 1_k \otimes w = \beta_l(w). \end{aligned}$$

This completes the proof of  $(\varepsilon \otimes \text{id})\Delta(w) = \beta_l(w)$  by induction.

By the same proof, we have  $(\text{id} \otimes \varepsilon)\Delta(w) = \beta_r(w)$ .  $\square$

By Lemmas 3.1-3.3, we have the following theorem.

**Theorem 3.4** *The quintuple  $(RS(X), \diamond, u, \Delta, \varepsilon)$  is a  $k$ -bialgebra.*

### 3.2 A Hopf algebra structure

For a  $k$ -algebra  $(A, \mu, u)$  and a  $k$ -coalgebra  $(C, \Delta, \varepsilon)$ , we define the convolution of two linear maps  $f, g \in \text{Hom}(C, A)$  to be the map  $f * g \in \text{Hom}(C, A)$  given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A.$$

Let  $(H, \mu, u, \Delta, \varepsilon)$  be a  $k$ -bialgebra. A  $k$ -linear endomorphism  $T$  of  $H$  is called an antipode of  $H$  if it is the inverse of  $\text{id}_H$  under the convolution product

$$T * \text{id}_H = \text{id}_H * T = u\varepsilon.$$

A Hopf algebra is a bialgebra  $H$  with an antipode  $T$ . For more about Hopf algebra, see for instance [1, 18].

Recall that a  $k$ -bialgebra  $(H, \mu, u, \Delta, \varepsilon)$  is called a graded bialgebra if there are a sequence of  $k$ -vector spaces  $H^{(n)}$ ,  $n \geq 0$ , such that

- (a)  $H = \bigoplus_{n=0}^{\infty} H^{(n)}$ ;
- (b) For any  $p, q \geq 0$ ,  $H^{(p)}H^{(q)} \subseteq H^{p+q}$ ;
- (c) For any  $p, q \geq 0$ ,  $\Delta(H^{(n)}) \subseteq \bigoplus_{p+q=n} H^{(p)} \otimes H^{(q)}$ .

A graded bialgebra  $H = \bigoplus_{n=0}^{\infty} H^{(n)}$  is called connected if  $H^{(0)} = k$ .

Define  $H_{RS}^{(n)}$  the  $k$ -linear space spanned by  $\{w \in \Phi_{\infty}(X) | \deg(w) = n\}$ , i.e.

$$H_{RS}^{(n)} = k\{w \in \Phi_{\infty}(X) | \deg(w) = n\}.$$

Then

$$H_{RS} = \bigoplus_{n=0}^{\infty} H_{RS}^{(n)}, \quad H_{RS}^{(0)} = k.$$

**Lemma 3.5** *Let  $p, q \geq 0$ . Then*

$$H_{RS}^{(p)} \diamond H_{RS}^{(q)} \subseteq H_{RS}^{(p+q)}.$$

**Proof.** We just have to prove that  $w \diamond v \in H_{RS}^{(p+q)}$  for any  $w, v \in \Phi_{\infty}(X)$  with  $\deg(w) = p$  and  $\deg(v) = q$ . Induction on  $n = p + q$ .

If  $n = 0$ , then  $w = v = 1$  and it is easy to see that the result is true.

Assume that the result is true for  $n = p + q$ .

Let  $w, v \in \Phi_{\infty}(X)$  with  $\deg(w) + \deg(v) = p + q + 1$ . Let

$$w = w_1 w_2 \cdots w_t, v = v_1 v_2 \cdots v_m.$$

Case 1. If  $(w_t, v_1) \notin \bigcup_{Q \in \{R, S\}} Q(\Phi_{\infty}(X)) \times Q(\Phi_{\infty}(X))$ , then  $w \diamond v = wv \in \Phi_{\infty}(X)$  and it is easy to see the result is true.

Case 2. If  $(w_t, v_1) \in \bigcup_{Q \in \{R, S\}} Q(\Phi_{\infty}(X)) \times Q(\Phi_{\infty}(X))$ , then  $w_t = R(w')$ ,  $v_1 = R(v')$  or  $w_t = S(w')$ ,  $v_1 = S(v')$ .

(i) If  $w_t = S(w')$ ,  $v_1 = S(v')$ , then

$$\begin{aligned} w \diamond v &= w_1 w_2 \cdots w_{t-1} (S(w') \diamond S(v')) v_2 \cdots v_m \\ &= w_1 w_2 \cdots w_{t-1} S(R(w') \diamond v' + w' \diamond S(v')) v_2 \cdots v_m. \end{aligned}$$

Let  $l = \deg(w_t) + \deg(v_1)$ . By induction hypotheses,  $R(w') \diamond v', w' \diamond S(v') \in H_{RS}^{(l-1)}$ . Therefore,  $w \diamond v \in H_{RS}^{(p+q+1)}$ .

(ii) If  $w_t = R(w')$ ,  $v_1 = R(v')$ , then similar to the proof of (i), we have the result is true.

This completes the proof by induction.  $\square$

**Lemma 3.6** *For any  $n \geq 0$ ,*

$$\Delta(H_{RS}^{(n)}) \subseteq \bigoplus_{p+q=n} H_{RS}^{(p)} \otimes H_{RS}^{(q)}.$$

**Proof.** For  $n = 0, 1$ , it is easy to see that the result is true. Assume that the result is true for  $0 \leq n \leq m$ . We just have to prove that  $\Delta(w) \in \bigoplus_{p+q=m+1} H_{RS}^{(p)} \otimes H_{RS}^{(q)}$  for any  $w$  with  $\deg(w) = m + 1$ .

If  $\text{bre}(w) = 1$ , then we have  $w = R(w')$  or  $w = S(w')$ .

If  $w = S(w')$ , then by induction hypotheses, we have

$$\Delta(w) = \Delta(S(w')) = R(w') \otimes 1 + (\text{id} \otimes S)\Delta(w') \in \bigoplus_{p+q=m+1} H_{RS}^{(p)} \otimes H_{RS}^{(q)}.$$

Similarly, if  $w = R(w')$ , then  $\Delta(w) \in \bigoplus_{p+q=m+1} H_{RS}^{(p)} \otimes H_{RS}^{(q)}$ .

If  $\text{bre}(w) \geq 2$ , say  $w = uv$  with  $u, v \neq 1$ , then by induction hypotheses and Lemma 3.5, we have

$$\begin{aligned} \Delta(w) &= \Delta(u) \diamond \Delta(v) \\ &\in \left( \bigoplus_{p+q=\deg(u)} H_{RS}^{(p)} \otimes H_{RS}^{(q)} \right) \diamond \left( \bigoplus_{p'+q'=\deg(v)} H_{RS}^{(p')} \otimes H_{RS}^{(q')} \right) \\ &\subseteq \bigoplus_{t+s=m+1} H_{RS}^{(t)} \otimes H_{RS}^{(s)}. \end{aligned}$$

This completes the proof by induction.  $\square$

**Lemma 3.7** ([18]) *A connected bialgebra  $(H, \mu, u, \Delta, \varepsilon)$  is a Hopf algebra.*

By Lemmas 3.5-3.7, we have the following theorem.

**Theorem 3.8** *The free Rota-Baxter system  $RS(X) = \bigoplus_{n=0}^{\infty} H_{RS}^{(n)}$  is a connected bialgebra. It follows that  $RS(X)$  is a Hopf algebra.*

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